CVIČENÍ Z MATEMATICKÉ ANALÝZY 2, 5. 6. 2020

$\boxed{1}$ $$y'' - y = \underbrace{\frac{e^x - e^{-x}}{e^x + e^{-x}}}_{f}$$

$y'' - y = 0 \qquad \lambda^2 - 1 = 0 \qquad \lambda^2 = 1 \qquad \lambda_1 = 1,\ \lambda_2 = -1$

F. S. $\quad e^{1\cdot x},\ e^{-1\cdot x} \qquad\qquad e^x,\ e^{-x}$

$y = c_1 e^x + c_2 e^{-x}$

$y' = \underbrace{c_1' e^x + c_2' e^{-x}}_{=0} + c_1 e^x - c_2 e^{-x}$

$y'' = c_1' e^x - c_2' e^{-x} + c_1 e^x + c_2 e^{-x}$

$$\begin{aligned} y'' - y &= c_1' e^x - c_2' e^{-x} + c_1 e^x + c_2 e^{-x} - c_1 e^x - c_2 e^{-x} \\ &= c_1' e^x - c_2' e^{-x} + c_1 \underbrace{(e^x - e^x)}_{=0} + c_2 \underbrace{(e^{-x} - e^{-x})}_{=0} \\ &= c_1' e^x - c_2' e^{-x} = f \end{aligned}$$

$$\begin{aligned} c_1' e^x + c_2' e^{-x} &= 0 \\ c_1' e^x - c_2' e^{-x} &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \end{aligned}$$

$c_1' = \dfrac{1}{2e^x} \dfrac{e^x - e^{-x}}{e^x + e^{-x}} \qquad c_2' = -c_1' e^x \cdot e^x = -\dfrac{1}{2} e^x \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$e^{-x} = t \qquad dx = -\frac{1}{t}\,dt$

$-e^{-x}\,dx = dt$

$$\int \frac{1}{2e^{x}} \cdot \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\,dx \overset{c}{=} \underbrace{\frac{1}{2}e^{-x} - \operatorname{arctg} e^{-x}}_{C_1}, \quad x \in \mathbb{R}$$

$$\int \left(\frac{1}{2} - \frac{1}{1+t^{2}}\right) dt \overset{c}{=} \frac{1}{2}t - \operatorname{arctg} t$$

$$\int -\frac{1}{2}e^{x}\, \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\,dx \overset{c}{=} \underbrace{-\frac{1}{2}e^{x} + \operatorname{arctg} e^{x}}_{C_2}$$

$e^{x} = t \qquad dx = \frac{1}{t}\,dt$

$e^{x}\,dx = dt$

$$y(x) = \left(\underline{\tfrac{1}{2}e^{-x}} - \operatorname{arctg} e^{-x}\right)\underline{e^{x}} + \left(\underline{-\tfrac{1}{2}e^{x}} + \operatorname{arctg} e^{x}\right)\underline{e^{-x}} + \alpha_1 e^{x} + \alpha_2 e^{-x}$$

$$y(x) = -e^{x}\operatorname{arctg} e^{-x} + e^{-x}\operatorname{arctg} e^{x} + \alpha_1 e^{x} + \alpha_2 e^{-x}, \quad x \in \mathbb{R}, \ \alpha_1, \alpha_2 \in \mathbb{R}$$

$\boxed{2}$ $y''+y=4x\cos x$

$y''+y=0 \qquad r^2+1=0 \qquad \lambda_1=i,\ \lambda_2=-i \qquad$ F.S.: $\sin x, \cos x$

$c_1'\sin x + c_2'\cos x = 0 \qquad / \cdot \sin x \qquad / \cos x$

$c_1'\cos x + c_2'(-\sin x) = 4x\cos x \qquad / \cdot \cos x \qquad / -\sin x$

$c_1' = 4x\cos^2 x \qquad c_2' = -4x\cos x\sin x = -2x\sin 2x$

$\int 4x\cos^2 x\,dx = \int 2x(1+\cos 2x)\,dx = x^2 + \int \underset{u}{2x}\,\underset{v'}{\cos 2x}\,dx$

$= x^2 + x\sin 2x - \int \sin 2x\,dx \overset{c}{=} x^2 + x\sin 2x + \frac{1}{2}\cos 2x$

$c_1 = x^2 + x\sin 2x + \frac{1}{2}\cos 2x$

$\int -2x\sin 2x\,dx = x\cos 2x - \int \cos 2x\,dx \overset{c}{=} x\cos 2x - \frac{1}{2}\sin 2x$

$c_2 = x\cos 2x - \frac{1}{2}\sin 2x$

$y(x) = (x^2 + x\underline{\sin 2x} + \frac{1}{2}\cos 2x)\cdot\underline{\sin x} +$

$\quad (x\,\underline{\cos 2x} - \frac{1}{2}\sin 2x)\,\underline{\cos x}$

$\quad + \alpha_1\sin x + \alpha_2\cos x$

$x(\sin 2x\cdot\sin x + \cos 2x\cdot\cos x) = x\cdot\cos x$

$\cos 2x\cdot\sin x - \sin 2x\cdot\cos x$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$y(x) = x^2\sin x + x\cos x + \beta_1\sin x + \beta_2\cos x,\ x\in\mathbb{R},\quad \beta_1,\beta_2\in\mathbb{R}$

$\boxed{3}$ $y''' - y'' - 2y' = e^{2x}$

$e^{\mu t}(P(t)\cos\nu t + Q(t)\sin\nu t)$

$t^m e^{\mu t}(\underline{R(t)}\cos\nu t + \underline{S(t)}\sin\nu t)$

$m \dots$ násobnost $\mu + i\nu$ kořene char. pol.

$\text{st}\, R \le \max\{\text{st}\, P, \text{st}\, Q\} = 0$

$\text{st}\, S \le \max\{\text{st}\, P, \text{st}\, Q\} = 0$

$\mu = 2 \quad \nu = 0 \quad Q(t) = 0 \quad P(t) = 1 \qquad \text{st}\, Q = -1$

$\text{st}\, P = 0$

$2 + 0i = 2$

$\lambda^3 - \lambda^2 - 2\lambda = 0$

$\lambda(\lambda^2 - \lambda - 2) = \lambda(\lambda - 2)(\lambda + 1) \quad \lambda_1 = 0, \lambda_2 = 2, \lambda_3 = -1$

F. S. $1, e^{2x}, e^{-x}$

$y(x) = x e^{2x} \cdot (c \cdot 1 + d \cdot 0) = x e^{2x} \cdot c$

$y'(x) = 2x e^{2x} c + e^{2x} c$

$y''(x) = 4x e^{2x} c + 2e^{2x} c + 2e^{2x} c = 4x e^{2x} c + 4e^{2x} c$

$y'''(x) = 8x e^{2x} c + 4e^{2x} c + 8e^{2x} c = 8x e^{2x} c + 12 e^{2x} c$

$y''' - y'' - 2y' = \underline{8}x e^{2x} c + \underline{12} e^{2x} c - (\underline{4} x e^{2x} c + \underline{4} e^{2x} c) - 2(\underline{2x} e^{2x} c + e^{2x} c)$

$= x e^{2x} c (8 - 4 - 4) + e^{2x} c (12 - 4 - 2) = e^{2x} c \cdot 6 = e^{2x} \Rightarrow c = \frac{1}{6}$

$y(x) = \frac{1}{6} x e^{2x} + \alpha_1 + \alpha_2 e^{2x} + \alpha_3 e^{-x}, \quad x \in \mathbb{R}, \quad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$

$\max\{\text{st}\, P, \text{st}\, Q\} = 2$